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On the steady motions produced by a stable stratification in a rapidly rotating fluid

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The equilibrium state of a rapidly rotating fluid, heated uniformly from above and cooled uniformly from below while contained in a cylinder with insulated side-walls is studied.

The circulations which are produced by the resulting stratification are studied over a wide range of parameters and it is shown that many of the features of the linear theory of rotating stratified fluid flows found in earlier studies reappear in this non-linear problem.

These include the gradual disappearance of Ekman layer suction and $O(1)$ Ekman layers as the stratification increases, and the determination of the interior flow by the side-wall boundary layers in conjunction with the Ekman layers.

It is suggested, therefore, that studies of rotating stratified flows which are unbounded laterally may frequently be defective and lead to solutions which are not the limit of any physically realizable experiment.

1. Introduction

The equilibrium state of a rotating fluid heated uniformly from *above* and cooled uniformly from *below* is the fundamental initial state from which many investigations of mechanically and/or thermally driven motions of a rotating, stratified fluid proceed. It is generally assumed, for example, that if the fluid is contained by vertically insulating walls, the resulting fluid state will be static relative to the rotating frame in which the container appears stationary. However, this cannot be so; for if the fluid is rotating as a rigid body with the angular velocity of the container, the isolines of constant density, pressure and temperature must be given by the equilibrium paraboloids, viz.

$$z - \Omega^2 r^2/2g = \text{constant},$$

where z and r are the co-ordinates respectively parallel and perpendicular to the rotation axis, while Ω and g are the magnitudes of the rotation and gravity vectors respectively, here assumed to be antiparallel. In this static state, the first law of thermodynamics becomes, with the usual approximations,

$$\nabla^2 T = 0.$$

There exists no solution of this equation for which the isotherms are equilibrium paraboloids. Therefore, either the temperature is constant, or the thermal diffusivity κ is identically zero, or, as must happen in a *real* stably stratified fluid, motions relative to the container's frame are produced (Greenspan 1967).

Since the equilibrium state is fundamental for both theoretical and experimental studies of the dynamics of rotating stratified fluids, it is of importance to understand completely its characteristics as a prerequisite to a full understanding of more complicated problems. Many theories proceed under the assumption that the centrifugal force is sufficiently small compared to the gravitational force so that the equilibrium paraboloids may be approximated by level surfaces. With this approximation, the basic stratification is linear and there is no resulting relative motion. Nevertheless, in any experiment some effect of the centrifugal force will be felt, and for rapidly rotating fluids the effect may not be negligible. The parameter which measures the deviation of the 'effective' gravitational potential from a level surface is the rotational Froude number,

$$F = \Omega^2 L/g,$$

where L is a characteristic dimension of the container.

In the present paper, we propose to examine which kind of stratification and motions are set up in a rotating fluid by a very simple vertical differential heating, when the effects of the Froude number are taken into account. We shall examine different régimes in which heat advection and/or conduction are the dominant processes which establish the basic stratification. Of course, in the advection dominated régime the heat equation is non-linear. The relative simplicity of the problem is, however, such that we shall be able to obtain certain important results. In fact, this provides a second motivation for the present analysis, namely, it will enable us to show that the results of our previous linear analysis (Barcilon & Pedlosky 1967*a*, *b*; hereafter referred to as B & P, I and II) of rotating, stratified fluids persist into the non-linear range. In particular, it appears that even in the non-linear régime, the diffusive processes are very important throughout the entire fluid region. Another important feature is the importance of the side walls in determining the fluid motions. When the fluid is stratified as well as rotating, the pre-eminence of the horizontal boundaries (perpendicular to the rotating axis), which is the central feature of the theory of homogeneous fluids, is lost. The constraint of the stratification, emphasizing information received by the fluid from the side walls can be as important as the rotational constraint which so strongly transmits information from horizontal boundaries. Theories which ignore this feature, leaving the fluid horizontally unbounded may be defective in the sense that they are not the limit of any physically realizable situation. An example of such a defective (but attractive) similarity solution is given in § 5.

2. Formulation

Let the fluid be contained in a cylinder of height L and radius R, which is rotating about its longitudinal axis with angular velocity Ω. The rotation axis is assumed anti-parallel to gravity. The sides of the cylinder are insulated while the top and bottom boundaries of the cylinder are maintained at fixed, constant temperatures $T_0 + \Delta T$ and T_0 respectively, where ΔT is a positive constant. For simplicity, the fluid is assumed to be incompressible, but viscous and heat conducting.

Recognizing the axial symmetry of the problem, we anticipate that all dependent variables will be functions only of r, the distance from the rotation axis, and z the vertical distance measured from the lower boundary. Let u, v and w be the radial, azimuthal and vertical velocity components respectively, while p, ρ, T are the symbols for pressure, density and temperature. We introduce the following non-dimensional variables, denoted by primes,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \frac{\alpha \Delta T g}{\Omega} \begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix},$$

$$\begin{Bmatrix} r \\ z \end{Bmatrix} = L \begin{Bmatrix} r' \\ z' \end{Bmatrix},$$

$$T = T_0 + (\Delta T)\, T',$$

$$p = \rho_0 g z - \rho_0 \tfrac{1}{2}(\Omega^2 r^2) + \rho_0 \alpha \Delta T g L p',$$

while the density is assumed to be related to the temperature by the simple state equation

$$\rho = \rho_0(1 - \alpha(\Delta T)\, T').$$

The parameter α is the (constant) coefficient of thermal expansion. The equations of motion, written in terms of the non-dimensional variables are, in the rotating frame (dropping the prime notation for the dimensionless variables)

$$(1-\epsilon FT)\,\epsilon\left(uu_r + wu_z - \frac{v^2}{r}\right) - 2v(1-\epsilon FT) = -p_r - FrT + (1-\epsilon FT)\,E\left[\nabla^2 u - \frac{u}{r^2}\right], \tag{2.1a}$$

$$\epsilon\left(uv_r + wv_z + \frac{uv}{r}\right) + 2u = E\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{2.1b}$$

$$(1-\epsilon FT)\,\epsilon(uw_r + ww_z) = -p_z + T + (1-\epsilon FT)\,E\nabla^2 w, \tag{2.1c}$$

$$(1/r)\,(ru)_r + w_z = 0, \tag{2.1d}$$

$$\sigma\epsilon(uT_r + wT_z) = E\nabla^2 T, \tag{2.1e}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Explicit use has been made of the axial symmetry. Four important dimensionless parameters enter into the problem; they are

$E = \nu/\Omega L^2$, the Ekman number,

$\epsilon = \alpha \Delta T g/\Omega^2 L$, the thermal Rossby number,

$\sigma = \nu/\kappa$, the Prandtl number,

and $F = \Omega^2 L/g$, the rotational Froude number.

The kinematic coefficient of viscosity is ν, the thermal diffusivity is κ. In this paper, asymptotic solutions will be presented for small ϵ and E, while σ and F will be considered $O(1)$ parameters. Naturally, in the limiting cases of small ϵ and E, the results may be expected to depend strongly on the relative magnitudes of ϵ and E as they formally approach zero in the asymptotic solutions.

The boundary conditions, which complete the specification of the problem are

$$T = \tfrac{1}{2} \pm \tfrac{1}{2} \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{2·2a}$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{on} \quad r = r_0 = R/L, \tag{2·2b}$$

$$(u, v, w) = 0 \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{2.2c}$$

and on $r = r_0$.

Note, that although in the limit of small ϵ the fluid satisfies the Boussinesq approximation (the density variations affecting the motion only through buoyancy effects) this, in general, includes buoyancy effects in the *radial as well as the vertical equations of motion*. The traditional Boussinesq approximation in which the buoyancy appears only in the vertical dynamics is valid only for small F. In fact, the radial component of effective gravity drives the motion in this case.

3. The linear solution

When the Rossby number, ϵ, is very small the non-linear terms may be neglected, yielding a relatively simple linear problem whose solution illustrates the essential mechanism which produces the circulations seen in the rotating frame.

Setting ϵ to zero, (2.1) becomes

$$-2v = -p_r - FrT + E(\nabla^2 u - u/r^2), \tag{3.1a}$$

$$2u = E(\nabla^2 v - v/r^2), \tag{3.1b}$$

$$0 = -p_z + T + E(\nabla^2 w), \tag{3.1c}$$

$$\frac{1}{r}(ru)_r + w_z = 0, \tag{3.1d}$$

$$0 = \nabla^2 T. \tag{3.1e}$$

From (3.1*e*) we note that in this conduction dominated limit, the problem for the temperature can be solved independently of the motion. The solution for T which satisfies (2.2*a, b*) is simply

$$T = z. \tag{3.2}$$

Since the temperature is therefore not constant along the isolines of the effective gravitational potential, motions will be produced. Note that the lines of constant temperature are *flat*, and are not curved in the shape of the equilibrium paraboloids. This is due to the presence of a small but non-zero thermal diffusivity of the fluid.

To obtain the velocity fields in the interior, outside of any boundary-layer region near solid surfaces, we may ignore the viscous terms proportional to E. This yields immediately the interior thermal wind relation for the azimuthal velocity,

$$2v_z = FrT_z + T_r, \tag{3.3}$$

which with (3.2) becomes

$$2v_z = Fr$$

or

$$v = \tfrac{1}{2}Frz + h(r), \tag{3.4}$$

where $h(r)$ must be determined.

We find similarly that

$$u = 0, \tag{3.5a}$$

$$w_z = 0; \tag{3.5b}$$

these relations are valid in the interior to $O(E)$.

The boundary conditions (2·2c) on $z = \frac{1}{2} \pm \frac{1}{2}$ are satisfied by the introduction of Ekman boundary layers of thickness $E^{\frac{1}{2}}$ on the horizontal rigid boundaries. Their dynamics are exactly the same as for homogeneous fluids and impose, as compatibility conditions on the interior flow, (B & P, I),

$$w = \mp \frac{E^{\frac{1}{2}}}{2r}\frac{\partial}{\partial r} rv \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{3.6}$$

These conditions, together with (3.5b) determine the function $h(r)$, i.e.

$$h(r) = -\tfrac{1}{4}Fr,$$

so that in the interior

$$v = \tfrac{1}{2}Fr(z - \tfrac{1}{2}), \tag{3.7a}$$

$$u = 0, \tag{3.7b}$$

$$w = -\tfrac{1}{4}F\,E^{\frac{1}{2}}. \tag{3.7c}$$

The interior azimuthal velocity is linear in z and antisymmetric about $z = \frac{1}{2}$ while on each horizontal plane the fluid rotates as a solid body. The vertical, interior velocity by which the fluid flows from the upper, hot plate to the lower cool plate is independent of r and z. The amplitude of the motion is proportional to the Froude number F. It is interesting to note in this limit of small temperature gradient, that the Ekman compatibility condition (3.6) completely determined the flow, as if the fluid were homogeneous.

The boundary condition (2.2c) on $r = r_0$ is satisfied through the introduction of a Stewartson layer of thickness $E^{\frac{1}{3}}$. The dynamics in this layer are again the standard homogeneous type originally described by Stewartson (1957). The resulting representations of the velocity fields, uniformly valid for all r outside the horizontal Ekman layers are

$$v = \tfrac{1}{2}F\left[r(z-\tfrac{1}{2}) + r_0 \sum_{n=1}^{\infty} \sin n\pi z \frac{(1-(-1)^n)}{n^2\pi^2}\left\{e^{-l_n\eta} + \frac{2}{\sqrt{3}}e^{-\frac{1}{2}l_n\eta}\sin\left(\frac{\sqrt{3}}{2}l_n\eta + \frac{\pi}{3}\right)\right\}\right], \tag{3.8a}$$

$$u = \frac{r_0E^{\frac{1}{3}}}{4}F\sum_{n=1}^{\infty}\cos n\pi z\frac{l_n^2(1-(-1)^n)}{n^2\pi^2}\left\{e^{-l_n\eta} - e^{-\frac{1}{2}l_n\eta}\cos\frac{\sqrt{3}}{2}l_n\eta + \frac{1}{2}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)e^{-\frac{1}{2}l_n\eta}\sin\frac{\sqrt{3}}{2}l_n\eta\right\}, \tag{3.8b}$$

$$w = \frac{r_0F}{4}\sum_{n=1}^{\infty}\frac{l_n^3\sin n\pi z\,(1-(1-)^n)}{(n^2\pi^2)}\left\{-e^{-l_n\eta} + \frac{2}{\sqrt{3}}e-{\tfrac{1}{2}l_n\eta}\sin\left(\frac{\sqrt{3}}{2}l_n\eta + \frac{\pi}{3}\right)\right\} - \frac{E^{\frac{1}{2}}F}{4}, \tag{3.8c}$$

where

$$\eta = (r_0 - r)\,E^{-\frac{1}{3}},$$

and

$$l_n = (2\pi)^{\frac{1}{3}}.$$

No layer of thickness $E^{\frac{1}{4}}$ is required since the interior azimuthal velocity has zero vertical average. Recalling that $F = O(1)$, it is now possible to delineate the region of validity of this linear solution.

In the interior, the vertical convection of heat, $\epsilon w T_z$, which is $O(\epsilon E^{\frac{1}{2}})$ becomes as important as the conduction when

$$\sigma\epsilon = O(E^{\frac{1}{2}}). \tag{3.9}$$

Similarly the buoyancy effects will become important in the $E^{\frac{1}{3}}$ layer, changing its character, when

$$\sigma\epsilon = O(E^{\frac{2}{3}}). \tag{3.10}$$

These are the critical values of the stratification deduced in B & P, II (called σS therein) for linear problems where the stratification was assumed known, not as here, to be determined. It is interesting to see them appear again. Since the *interior* solution is given by (3.7) until $\sigma\epsilon = O(E^{\frac{1}{2}})$ we shall concentrate our attention on the character of the solution as that critical value is attained and surpassed. The reader is referred to B & P, II, for a discussion of the interesting metamorphosis of the side-wall layers as the critical stratification

$$\sigma\epsilon = O(E^{\frac{2}{3}})$$

is reached (which occurs first).

We turn our attention next to the case where thermal convection is of equal importance in the interior as the thermal conduction and therefore where non-linear effects are important, i.e. when $\sigma\epsilon = O(E^{\frac{1}{2}})$.

4. The effect of convection; $\sigma\epsilon = O(E^{\frac{1}{2}})$

4.1. *The interior problem*

To investigate the nature of the solution when the effect of thermal convection becomes important it is convenient to introduce explicitly a relation between $\sigma\epsilon$ and E. Assuming $\sigma = O(1)$, we write

$$\epsilon = \lambda E^{\frac{1}{2}}, \tag{4·1}$$

where λ is an $O(1)$ constant.

To determine the interior equations of motion, (4.1) is inserted into (2.1), and the variables appearing in the resulting equations are expanded in the following asymptotic series, presumed valid outside boundary-layer regions.

$$\left.\begin{aligned} u &= Eu_2 + \ldots, \\ v &= v_0 + E^{\frac{1}{2}}v_1 + \ldots, \\ w &= E^{\frac{1}{2}}w_1 + \ldots, \\ T &= T_0 + E^{\frac{1}{2}}T_1 + \ldots, \\ p &= p_0 + E^{\frac{1}{2}}p_1 + \ldots. \end{aligned}\right\} \tag{4.2}$$

Inserting (4.2) into the equations of motion and expanding the result in powers of $E^{\frac{1}{2}}$ we obtain

$$2v_0 = \frac{\partial p_0}{\partial r} + rT_0 F, \tag{4.3a}$$

$$0 = -\frac{\partial p_0}{\partial z} + T_0, \tag{4.3b}$$

$$0 = \frac{\partial w_1}{\partial z}, \tag{4.3c}$$

$$\sigma\lambda w_1 T_{0z} = \nabla^2 T_0 \tag{4.3d}$$

and the derived thermal wind relation

$$2\frac{\partial v_0}{\partial z} = \frac{\partial T_0}{\partial r} + Fr\frac{\partial T_0}{\partial z}. \tag{4.3e}$$

The non-linear nature of the problem is revealed by (4.3*d*), which serves as the determining equation of motion. The vertical convection of heat alters the purely conductive state derived in §3. Since w_1 is independent of z, this vertical convection is accomplished by the Ekman layer suction velocity, which as before is given by the relation

$$w_1 = \mp\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}(rv_0) \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{4.4a}$$

obtained by studying the Ekman layers needed to satisfy the kinematic boundary conditions (2.2*c*) on the horizontal bounding surfaces. Another consequence of a perusal of the characteristics of the Ekman layer (which is the only possible horizontal boundary layer in this parameter region) is that the Ekman layer corrections to the temperature field are at most of $O(E)$, so that the interior temperature must satisfy the conditions

$$T_0 = \tfrac{1}{2} \pm \tfrac{1}{2} \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{4.4b}$$

Integrating (4.3*e*) from $z = 0$ to $z = 1$, yields, with the application of (4.4*a*) and (4.4*b*)

$$w_1 = -\frac{1}{4}\left[F + \tfrac{1}{2}D^2\int_0^1 T_0\,d\dot{z}\right], \tag{4.5}$$

where
$$D^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}.$$

The governing equation (4.3*d*), rewritten in terms of only the temperature, is

$$\nabla^2 T_0 + \frac{\sigma\lambda}{4}\left[F + \tfrac{1}{2}D^2\int_0^1 T_0\,dz\right]T_{0z} = 0. \tag{4.6}$$

To complete the problem, boundary conditions for (4.6) to be applied on $r = r_0$, must be deduced by matching the boundary condition (2.2*b*) to the interior through the application of the dynamics of the side-wall boundary layers.

4.2. *The side-wall layers*

The side-wall boundary layer, in the case when $\sigma\epsilon = O(E^{\frac{1}{2}})$ has a double structure, similar to that described in B & P, II (a triple structure was in general observed there, two of the sublayers merging when the equivalent condition $\sigma S = O(E^{\frac{1}{2}})$ occurred). The outer layer has a thickness of $O(E^{\frac{1}{4}})$ (a combination of the $E^{\frac{1}{4}}$ and hydrostatic layers found in B & P, II), and is a baroclinic generalization of the Stewartson $E^{\frac{1}{4}}$ layer, while the inner sublayer has a thickness of

$$O(E^{\frac{1}{2}}\epsilon^{-\frac{1}{4}}) = O(E^{\frac{3}{8}}),$$

and is a generalization of the buoyancy layer described in B & P, II.

In the outer layer the representation of the dependent variables is

$$\left.\begin{aligned} v &= v_0(r_0,z)+\bar{v}(\eta,z)+\ldots,\\ u &= E^{\frac{1}{2}}\bar{u}(\eta,z),\\ w &= E^{\frac{1}{4}}\bar{w}(\eta,z)+\ldots,\\ T &= T_0(r_0,z)+E^{\frac{1}{4}}\bar{T}(\eta,z)\ldots,\\ p &= p_0(r_0,z)+E^{\frac{1}{4}}\bar{p}\,(\eta,z)+\ldots,\end{aligned}\right\} \tag{4.7}$$

where
$$\eta = (r_0-r)\,E^{-\frac{1}{4}},$$

and the boundary-layer corrections to the interior flow are denoted by bars.

Inserting (4.7) into (2.1), the equations for the boundary-layer corrections which must vanish as $\eta\to\infty$ are

$$\left.\begin{aligned} 2\bar{v} = -\bar{p}_\eta \quad 2\bar{u} = \bar{v}_{\eta\eta}, \quad \bar{p}_z = \bar{T},\\ \bar{u}_\eta = \bar{w}_z, \quad \sigma\lambda\bar{w}T_{0z}(r_0,z) = \bar{T}_{\eta\eta}.\end{aligned}\right\} \tag{4.8}$$

Eliminating all variables in favour of $\bar{v}$ we obtain, as the governing boundary-layer equation

$$\frac{\partial^2\bar{v}}{\partial\eta^2}+\frac{\partial}{\partial z}\frac{1}{l^4}\frac{\partial\bar{v}}{\partial z} = 0, \tag{4.9}$$

where
$$l^4(z) = \tfrac{1}{4}\sigma\lambda T_{0z}(r,z).$$

The boundary conditions for (4.9) on $z = 0$ and 1 can be found by applying the Ekman layer compatibility condition (3.6) to the boundary-layer variables. This is a valid procedure because the Ekman layer is much thinner than the $E^{\frac{1}{4}}$ layer. This yields

$$\frac{\partial\bar{v}}{\partial z} = \mp\, l^4\bar{v} \quad \text{on} \quad z = \tfrac{1}{2}\pm\tfrac{1}{2}. \tag{4.10}$$

To solve (4.9) subject to (4.10) we write

$$\bar{v} = \sum_{n=0}^{\infty} C_n e^{-k_n\eta}\,\bar{V}_n(z), \tag{4.11}$$

which produces the Sturm–Liouville problem for $(\bar{V}_n, k_n)$:

$$\frac{d}{dz}\frac{1}{l^4}\frac{d}{dz}\bar{V}_n+k_n^2\bar{V}_n = 0, \tag{4.12}$$

$$\frac{d\bar{V}_n}{dz} = \mp\, l^4\bar{V}_n \quad \text{on} \quad z = \tfrac{1}{2}\pm\tfrac{1}{2}. \tag{4.13}$$

As long as $l^4(z) > 0$, which implies that the fluid is stably stratified, the eigenvalue problem is non-singular and by the usual Sturm–Liouville theorems (4.12) and (4.13) possess a complete set of solutions, which can be used to represent any function of z. It can also be shown that

$$k_n^2 = \frac{\displaystyle\int_0^1 l^{-4}\left(\frac{d\overline{V}_n}{dz}\right)^2 dz + \overline{V}_n^2(1) + \overline{V}_n^2(0)}{\displaystyle\int_0^1 \overline{V}_n^2\, dz} > 0, \tag{4.14}$$

so that all solutions to (4.9) are exponential in η and of boundary-layer type. In terms of the solutions of (4.12) (which cannot be explicitly determined until $T_{0z}(r_0, z)$ is known, i.e. the interior problem solved) the boundary-layer variables may be written

$$\left.\begin{aligned} \bar{v} &= \sum_{n=0}^{\infty} C_n e^{-k_n\eta}\, \overline{V}_n(z), \\ \bar{u} &= \frac{1}{2}\sum_{n=0}^{\infty} C_n e^{-k_n\eta} k_n^2\, \overline{V}_n(z), \\ \overline{T} &= 2\sum_{n=0}^{\infty} C_n k_n^{-1} \frac{d\overline{V}_n}{dz} e^{-k_n\eta}, \\ \overline{w} &= \frac{l^{-4}}{2}\sum_{n=0}^{\infty} C_n k_n \frac{d\overline{V}_n}{dz} e^{-k_n\eta}, \end{aligned}\right\} \tag{4.15}$$

where, naturally, the positive square root of the eigenvalue k_n^2 is used to define k_n. The constants C_n will be determined in terms of the interior azimuthal velocity by matching the side-wall condition $v = v_0 + \bar{v} = 0$ on $r = r_0$.

It will be convenient to have explicitly a relation for the stream function $\overline{\psi}$ of the boundary-layer meridional motion. Since $r_0\bar{u} = \overline{\psi}_z$ and $r_0\overline{w} = \overline{\psi}_\eta$ this can be obtained from (4.15):

$$\overline{\psi} = -\tfrac{1}{2} r_0 l^{-4} \sum_{n=0}^{\infty} C_n \frac{dV_n}{dz} e^{-k_n\eta}. \tag{4.16}$$

Within the inner side-wall layer, the bouyancy layer,† where $(r - r_0) = O(E^{\frac{3}{8}})$, another correction to the field representations is required. In this region we write

$$\left.\begin{aligned} v &= v_0 + \bar{v} + \ldots + E^{\frac{1}{4}}\hat{v}(x,z) + \ldots, \\ u &= E^{\frac{1}{2}}\bar{u} + \ldots + E^{\frac{1}{2}}\hat{u}(x,z) + \ldots, \\ w &= E^{\frac{1}{8}}\hat{w}(x,z) + \ldots, \\ T &= T_0(r_0, z) + E^{\frac{1}{4}}\overline{T} + \ldots + E^{\frac{3}{8}}\hat{T}(x,z) + \ldots, \\ p &= p_0(r_0, z) + E^{\frac{1}{4}}\bar{p} + \ldots + E^{\frac{5}{8}}\hat{p}(x,z) + \ldots, \end{aligned}\right\} \tag{4.17}$$

where the caret denotes the boundary-layer corrections in the buoyancy layer which must vanish as

$$x = (r_0 - r)\, E^{-\frac{3}{8}}$$

becomes large.

† This terminology, which was used in our previous paper, is due to Veronis (private communication).

By inserting (4.17) into (2.1), equations for the buoyancy layer correction functions are obtained, i.e.

$$\left.\begin{aligned} 2\hat{v} &= -\hat{p}_x, \\ 2\hat{u} &= \hat{v}_{xx}, \\ 0 &= \hat{T} + \hat{w}_{xx}, \\ \sigma\lambda\hat{w}T_{0z}(r_0, z) &= \hat{T}_{xx}. \end{aligned}\right\} \tag{4.18}$$

These equations are essentially the same as those found in B & P for the case of the linear dynamics, with the important exception that now $T_0(r_0, z)$ is not explicitly known. Nevertheless, it is important to note that the qualitative nature of the dynamics of the side-wall layer found in the linear problem also holds in this non-linear case. The set of equations (4.18) may be solved subject to the boundary condition

$$\hat{w}(0) = 0.$$

This is required since $w = 0$ on $r = r_0$, and the buoyancy layer correction to w swamps the $E^{\frac{1}{4}}$ and interior contributions to w at the wall. The solutions for the buoyancy-layer corrections are

$$\hat{T} = -\frac{4l^3}{r_0} A_0(z)\,[\cos lx + \sin lx]\,e^{-lx}, \tag{4.19a}$$

$$\hat{w} = -\frac{2l}{r_0} A_0(z) \sin lx\, e^{-lx}, \tag{4.19b}$$

$$\hat{u} = \frac{1}{r_0}\frac{d}{dz}[A_0(z)\, e^{-lx}(\cos lx + \sin lx)], \tag{4.19c}$$

while the correction to the stream function $\hat{\psi}$ for the meridional motion is

$$\hat{\psi} = A_0\, e^{-lx}\,(\cos lx + \sin lx), \tag{4.19d}$$

where $\hat{u} = r_0^{-1}\hat{\psi}_z, \quad \hat{w} = r_0^{-1}\hat{\psi}_x, \quad \text{and} \quad l^4 = \tfrac{1}{4}\sigma\lambda T_{0z}(r_0, z).$

To determine the boundary condition on the interior flow at $r = r_0$, as well as the C_n's and A_0, we must now apply the side-wall boundary conditions. The no-slip condition on w has already been satisfied by the buoyancy layer. The no-slip conditions on the azimuthal velocity yields

$$v_0 + \bar{v} = 0 \quad \text{on} \quad r = r_0,$$

or

$$v_0(r_0, z) + \sum_{n=0}^{\infty} C_n \bar{V}_n(z) = 0 \quad \text{on} \quad r = r_0,$$

i.e.

$$C_n = -\frac{\int_0^1 v_0(r_0, z)\,\bar{V}_n(z)\,dz}{\int_0^1 \bar{V}_n^2\,dz} \tag{4.20}$$

so that once v_0 is known, i.e. the interior problem solved, C_n is determined.

The condition $u = 0$ on $r = r_0$ is more conveniently satisfied as the requirement that $r = r_0$ be a streamline of the total meridional motion, i.e. that

$$\psi_I + \bar{\psi} + \hat{\psi} = 0 \quad \text{on} \quad r = r_0, \tag{4.21}$$

where

$$\psi_I = \int_0^r rw\,dr.$$

The condition that the side-wall be insulated is

$$T_{0r} - \overline{T}_\eta - \hat{T}_x = 0 \quad \text{on} \quad r = r_0. \tag{4.22}$$

By integrating the heat equation across the side-wall layer, it can be shown that

$$\overline{T}_\eta + \hat{T}_x = \frac{4l^4}{r_0}(\overline{\psi} + \hat{\psi}) \quad \text{on} \quad r = r_0. \tag{4.23}$$

Thus on $r = r_0$, we find, using (4.21) and (4.22), the boundary condition for the interior flow

$$\frac{4l^4}{r_0}\psi_I + T_{0r} = 0 \quad \text{on} \quad r = r_0. \tag{4.24}$$

Since
$$\psi_I(r_0) = \int_0^{r_c} rw_1\,dr,$$

(4.5) may be used to express (4.24) entirely in terms of the temperature field, viz.

$$\frac{\partial T_0}{\partial r} = -\frac{r_0 F\sigma\lambda}{(\sigma\lambda+8)}\frac{\partial T_0}{\partial z}, \tag{4.25}$$

which is the most suitable form for the side-wall boundary condition for T_0. Equation (4.7), together with (4.4*b*) and (4.25) sets the problem for T_0. No general solution to this problem has been found, but an approximate solution, valid for $F < 1$ (but much greater than $E^{\frac{1}{2}}$) can be found as follows.

For small F let

$$T_0 = z + F\theta,$$

Then θ satisfies the following equation and boundary conditions to $O(F)$

$$\nabla^2\left[\theta + \frac{\sigma\lambda}{8}\int_0^1 \theta\,dz\right] = -\frac{\sigma\lambda}{4}, \tag{4.26a}$$

$$\theta = 0 \quad \text{on} \quad z = 0, 1, \tag{4.26b}$$

$$\theta_r = -\frac{\sigma\lambda r_0}{(8+\sigma\lambda)} \quad \text{on} \quad r = r_0. \tag{4.26c}$$

If we define
$$\tau = \theta + \frac{\sigma\lambda}{8}\int_0^1 \theta\,dz,$$

then
$$\nabla^2\tau = -\frac{\sigma\lambda}{4}, \tag{4.27a}$$

$$\tau_r = -\frac{\sigma\lambda r_0}{8} \quad \text{on} \quad r = r_0,$$

$$\tau - \frac{\sigma\lambda}{(\sigma\lambda+8)}\int_0^1 \tau\,dz = 0 \quad \text{on} \quad z = 0, 1. \tag{4.27b}$$

Once τ is found, θ may be obtained from the relation

$$\theta = \tau - \frac{\sigma\lambda}{\sigma\lambda+8}\int_0^1 \tau\,dz. \tag{4.28}$$

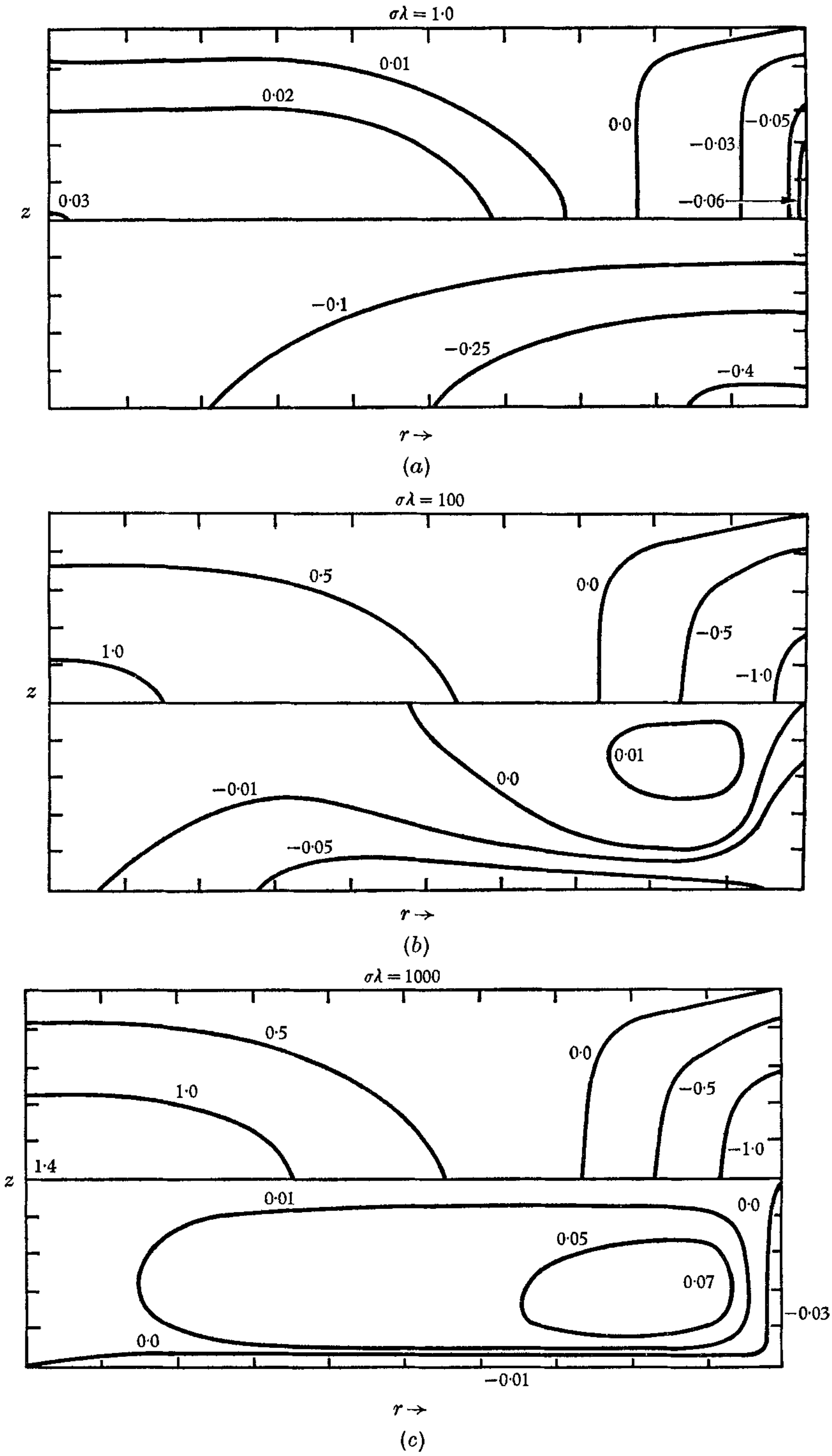

FIGURE 1. The upper portion of the figure shows lines of constant θ in the region $\frac{1}{2} \leqslant z \leqslant 1$. The lower portion of the figure shows lines of constant $v_0 F^{-1}$ in the region $0 \leqslant z \leqslant \frac{1}{2}$. Note that θ is symmetric about $z = \frac{1}{2}$ while v_0 is antisymmetric. (a) $\sigma\lambda = 1$, (b) $\sigma\lambda = 100$, (c) $\sigma\lambda = 1000$. r_0 is taken equal to 2.

The solutions for θ, v_0 and w_1 can easily be found to be

$$\theta = \frac{\sigma\lambda}{\sigma\lambda+8}\frac{r_0^2}{2}\left[\frac{1}{2}-\left(\frac{r}{r_0}\right)^2+4\sum_{n=1}^{\infty}\frac{1}{k_n^2}\frac{J_0(k_n r/r_0)}{J_0(k_n)}\left\{\frac{\cosh\frac{k_n}{r_0}(z-\frac{1}{2})-\frac{\sigma\lambda}{\sigma\lambda+8}\frac{2r_0}{k_n}\sinh\frac{k_n}{2r_0}}{\cosh\frac{k_n}{2r_0}-\frac{\sigma\lambda}{\sigma\lambda+8}\frac{2r_0}{k_n}\sinh\frac{k_n}{2r_0}}\right\}\right], \tag{4.29}$$

$$v_0 = F\frac{r}{2}(z-\tfrac{1}{2})\frac{8}{\sigma\lambda+8} - \frac{\sigma\lambda F r_0^2}{\sigma\lambda+8}\sum_{n=1}^{\infty}\frac{J_1(k_n r/r_0)}{k_n^2 J_0(k_n)}\left\{\frac{\sinh\frac{k_n}{r_0}(z-\frac{1}{2})-\frac{\sigma\lambda}{\sigma\lambda+8}2(z-\frac{1}{2})\sinh\frac{k_n}{2r_0}}{\cosh\frac{k_n}{2r_0}-\frac{\sigma\lambda}{\sigma\lambda+8}\frac{2r_0}{k_n}\sinh\frac{k_n}{2r_0}}\right\}, \tag{4.30}$$

and

$$w_1 = -\frac{F}{4}\left(\frac{8}{\sigma\lambda+8}\right)+\frac{4\sigma\lambda}{(\sigma\lambda+8)^2}Fr_0\sum_{n=1}^{\infty}\frac{J_0\left(k_n\frac{r}{r_0}\right)\sinh\frac{k_n}{2r_0}}{k_n J_0(k_n)\left[\cosh\frac{k_n}{2r_0}-\frac{\sigma\lambda}{\sigma\lambda+8}\frac{2r_0}{k_n}\sinh\frac{k_n}{2r_0}\right]}. \tag{4.31}$$

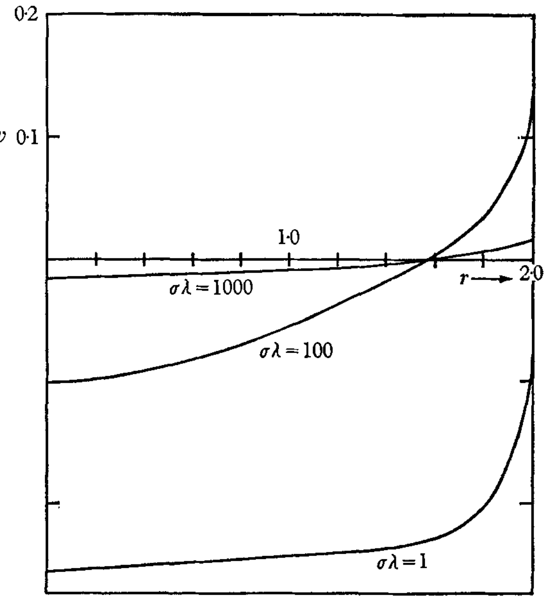

FIGURE 2. The interior vertical velocity, w_1, for (*a*) $\sigma\lambda = 1\cdot0$, (*b*) $\sigma\lambda = 100$, (*c*) $\sigma\lambda = 1000$.

The series for θ, v_0 and w_1 were summed numerically; lines of constant θ and v_0 for values of $\sigma\lambda$ equal to 1·0, 100 and 1000 are displayed in figure 1. The vertical velocity for these values of $\sigma\lambda$ is displayed in figure 2. Figure 1*a* corresponds essentially to the conduction dominated régime. As the advection becomes more important the temperature profile is distorted by the vertical motion and the temperature field acquires a more pronounced radial structure, as implied by the condition (4.25) In fact the cold fluid which is flushed up through the side-wall layers cools the interior slightly, especially near the side-walls, as is evident from figures 1*b* and 1*c*. Note that the $O(1)$ azimuthal velocity remains antisymmetric about $z = \frac{1}{2}$, but its radial and vertical structure becomes more complicated in response to the cool side-wall layers which reduce the azimuthal velocity by thermal wind effects. It is also interesting to note (figure 2) how the magnitude of the vertical, interior velocity is reduced with increasing $\sigma\lambda$, i.e. increasing stable stratification.

Now that v_0 and T_0 are known the side-wall layer constants C_n and the function $A_0(z)$ can be determined. For the sake of brevity they are not presented here.

5. The convection dominated régime $\sigma\epsilon \gg E^{\frac{1}{2}}$

It would appear that as the Rossby number ϵ is increased the effect of the thermal convection would dominate the effects of thermal conduction. In fact this is not the case; *thermal conduction remains important throughout the fluid for all* ϵ. Before discussing this surprising result in more detail, it is of interest to note the existence of a similarity solution to (2.1) which is valid over a wide range of $\sigma\epsilon$,

Writing

$$w = E^{\frac{1}{2}}w_1 + \ldots, \quad u = O(E), \quad T = T_0 + \ldots, \quad p = p_0 + \ldots,$$

and *specifying* that w_1 is a constant and T_0 is independent of r we find

$$T_{0zz} = w_1 \frac{\sigma\epsilon}{E^{\frac{1}{2}}} T_{0z},$$

yielding

$$T_0 = \frac{1-e^{-\beta z}}{1-e^{-\beta}}, \tag{5.1a}$$

where $\beta = -w_1\sigma\epsilon/E^{\frac{1}{2}}$. Matching the Ekman compatibility conditions yields

$$w_1 = -\tfrac{1}{4}F, \tag{5.1b}$$

$$v_0 = \frac{2}{r}F\left[\frac{1-e^{-\beta z}}{1-e^{-\beta}} - \frac{1}{2}\right]. \tag{5.1c}$$

This similarity solution, valid outside the Ekman layers is a possible solution to the complete problem in the absence of any containing side-wall boundary, and is similar to the one found in a slightly different context by Duncan (1966). The solution has the following interesting properties. For small $\sigma\epsilon/E^{\frac{1}{2}}$ the similarity solution (5.1) reduces to the interior solution found in §3. For large $\sigma\epsilon/E^{\frac{1}{2}}$ the solution becomes more asymmetric in z and finally achieves a state wherein the bulk of the region $0 \leqslant z \leqslant 1$ consists of a homogeneous fluid with a temperature equal to that of the upper boundary which is swept down to within a distance of $O(E^{\frac{1}{2}}/\sigma\epsilon)$ of the lower boundary where a thermal boundary layer (where conduction is important) matches this homogeneous interior to the lower boundary temperature. The azimuthal velocity shares a similar asymmetric distribution in z for large $\sigma\epsilon/E^{\frac{1}{2}}$. Thus for large $\sigma\epsilon/E^{\frac{1}{2}}$ the similarity solution is convection dominated.

The similarity solution is at once so simple and intuitive that several futile attempts were made by us to use it as an interior solution and effectively enclose it in a finite cylinder by adding appropriate side-wall boundary layers. It became clear, however, that the *presence* of the side-wall boundaries eliminated the relevance, not only of the details of the solution, but that the dynamics of the motion in a closed region was essentially different from that inferred from the similarity solution. The similarity solution is not the limiting solution of a physically realizable experiment in a closed container, even with a large radius.

There are many ways of seeing this. First, it is not formally possible to join the similarity solution to the side-wall boundary condition through the use of side-wall layers. Alternately, it may be noted that in a finite region the fluid *must return* through the side-wall layers and the intense stable stratification produced in the thermal layers near $z = 0$ of the similarity solution effectively prevents this, choking off the meridional circulation required by the similarity solution. In

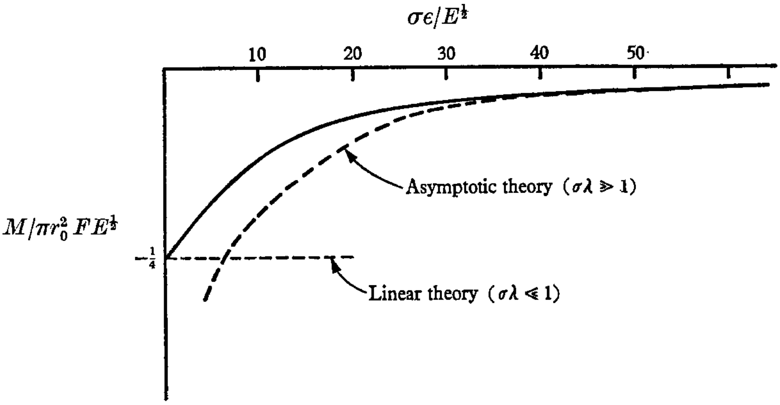

FIGURE 3. ——, vertical interior mass transport; – – –, results from the linear and the asymptotic analysis.

addition, in a closed region insulated at the side-walls, the total heat flux, conductive plus convective, must be independent of z. The similarity solution implies that the upper boundary covers essentially homogeneous fluid, resulting in the contradiction that no heat is flowing from $z = 1$, where $T = 1$ to $z = 0$, where $T = 0$. For all these reasons, the similarity solution is defective and dynamical conclusions drawn from it are generally invalid, primarily because the effects of the side-wall layers are ignored, and by choking off the meridional flow they control the dynamics of the entire interior. It is important to note that if the similarity solution were correct, each streamline would pass through the side-wall layers.

Finally, by integrating (2.1*e*) over the volume of the fluid and explicitly balancing the net heat flux on the planes at $z = 1$ and $z = 0$ we find that the total vertical interior flux M is

$$M = 2\pi E^{\frac{1}{2}} \int_0^{r_0} r w_1 \, dr = -2\pi r_0^2 \frac{FE^{\frac{1}{2}}}{(8+\sigma\epsilon/E^{\frac{1}{2}})}. \tag{5.2}$$

For large $\sigma\epsilon/E^{\frac{1}{2}}$ this flux falls to $O(E/\sigma\epsilon)$ which is much smaller than that needed for the validity of the similarity solution (see figure 3).

A consistent dynamical picture for the case $\sigma\epsilon \gg E^{\frac{1}{2}}$ emerges then, as follows. As opposed to the similarity scaling we write, for the interior variables

$$\left.\begin{aligned} w = E/\sigma\epsilon w_1, \quad v = v_0 + \ldots, \quad u = O(E), \\ T = T_0 + \ldots, \quad p = p_0 + \ldots. \end{aligned}\right\} \tag{5.3}$$

Inserting (5.3) into (2.1) yields

$$2v_0 = p_{0r} + rFT_{0z}, \tag{5.4a}$$

$$0 = p_{0z} + T_0, \tag{5.4b}$$

$$w_1 T_{0z} = \nabla^2 T_0, \tag{5.4c}$$

$$w_{1z} = 0. \tag{5.4d}$$

Using (5.4d) we obtain the governing equation by differentiating (5.4c) with respect to z, and then using (5.4b) we find that

$$p_{0zz}\nabla^2 p_{0zz} - \nabla^2 p_{0z} p_{0zzz} = 0, \tag{5.5}$$

which is the non-linear generalization of the equation found in B & P, I, for the case of substantial stratification. The boundary conditions on $z = 0$ and 1 are of interest. Since the interior vertical velocity is $O(E/\epsilon) \ll E^{\frac{1}{2}}$, the Ekman compatibility condition, (3.6) requires that

$$\frac{1}{r}\frac{\partial}{\partial r} rv_0 = 0 \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{5.6}$$

Since the region is simply connected, no non-singular solution of (5.6) exists except the solution

$$v_0 = 0 \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}.$$

The interior solution therefore satisfies the no slip condition on $z = 0, 1$ and no $O(1)$ Ekman layers are required. This was one of the central results we found in the *linear* dynamics of heavily stratified fluids. It is interesting to note that the non-linearity, which allows, *a priori*, the fluid to select its own local stratification, does not alter this important result. Rather, the over-all stratification is essentially found *throughout* the fluid, reducing the magnitude of the secondary circulations, re-introducing the effects of dissipation into the interior, even for substantial values of the Rossby number.

Thus

$$p_{0z} = \tfrac{1}{2} \pm \tfrac{1}{2} \quad \text{on} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{5.7a}$$

$$p_{0r} = 0 \quad \text{on} \quad z = 0, \tag{5.7b}$$

$$p_{0r} = -rF \quad \text{on} \quad z = 1. \tag{5.7c}$$

Proper conditions on the interior problem on $r = r_0$ are found by considering the dynamics of the side-wall boundary layers. Their structure is similar to the linear examples in (B & P, II) and the layers displayed in §4.

There is an inner buoyancy layer of thickness $E^{\frac{1}{2}}\epsilon^{-\frac{1}{4}}$ in which the vertical transport matches the interior flux, and a baroclinic, hydrostatic layer with a thickness $\epsilon^{\frac{1}{2}}$. The dynamics of the latter is the same as the $E^{\frac{1}{4}}$ layer described in §4, and in fact coincides with it when $\sigma\epsilon = O(E^{\frac{1}{2}})$. For the sake of brevity, and because the dynamics of the layer is essentially the same as in the case $\sigma\epsilon = O(E^{\frac{1}{2}})$ the details are omitted.

The boundary condition on the interior flow derivable from matching the interior flow to the side-wall conditions through the side-wall layers can be written as

$$\frac{\partial T_0}{\partial r}(r_0, z) = \left[\frac{1}{r_0}\int_0^{r_0} rw_1\, dr\right]\frac{\partial T_0}{\partial z}(r_0, z) \quad \text{on} \quad r = r_0 \tag{5.8}$$

and is similar to (4.24). Using (5.2) (after noting the rescaling of w), in the limit of large $\sigma\epsilon/E^{\frac{1}{2}}$, we obtain

$$\frac{\partial T_0}{\partial r} = -r_0 F\frac{\partial T_0}{\partial z} \quad \text{on} \quad r = r_0, \tag{5.9}$$

to which (4.25) tends for large $\sigma\lambda$. Thus the equilibrium paraboloidal slope is attained on $r = r_0$ for the interior solution for large $\sigma\epsilon/E^{\frac{1}{2}}$. In terms of p_0 this condition is

$$\frac{\partial^2 p_0}{\partial z\, \partial r} = -r_0 F\frac{\partial^2 p_0}{\partial z^2} \quad \text{on} \quad r = r_0, \tag{5.10}$$

which completes the specification of the interior problem.

Although the dynamical nature of the flow is clear the detailed solution of (5·5) subject to (5·6) and (5·10) is analytically intractable. For the purpose of obtaining detailed results we once again investigate the case of moderately small F. It is not necessary to produce any detailed calculations, for it was found that the solutions obtained were directly obtainable from (4.29) and (4.30) by letting $\sigma\lambda \to \infty$. The interior solution is therefore continuous in the parameter $\sigma\epsilon/E^{\frac{1}{2}}$ which again is a result qualitatively similar to what we found in (B & P, II), and *completely* different from the situation suggested by the similarity solution. Finally, it is interesting to note that the various physical fields exhibit certain symmetries as opposed to the similarity solution.

6. Conclusions

When a rotating fluid is heated uniformly from above and cooled uniformly from below, steady circulations are produced in response to the departure of the isolines of the 'effective' gravitational potential from the isotherms, whose configuration is determined to a large degree by thermal diffusion.

It was shown that the presence of the side-walls plays an important role in determining the flow once the imposed stratification becomes sufficiently large. In this limit the rotational constraint and the Ekman layer suction no longer dominate the flow.

Further, although through non-linear effects the fluid may adjust its own stratification locally, the nature of the dynamics, is, in each parameter region investigated, essentially the same as the linear dynamics found by us when the stratification was imposed throughout. This suggests that many features of the general linear theory may be observed in experiments in which non-linear effects are not negligible. One of the most important of these features is the disappearance of the Ekman layer in a substantially stratified fluid.

We have both benefited greatly from our close association with Harvey P. Greenspan, who has stimulated our interest in many problems of Rotating Fluid Dynamics of which this is one. It is a pleasure to acknowledge his influence. This research was partially supported by the Office of Scientific Research of the U.S. Air Force, Grant AF-AFOSR-492-66.

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